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Math 564: Adv. Analysis 1 HOMEWORK 2 **Due: Sep 29 (Fri), 11:59pm**

1. Let μ be a finite premeasure on an algebra $\mathcal{A}$ on a set X. Recall from Carathéodory's theorem that μ^* is an extension of μ to a measure on $\langle \mathcal{A} \rangle_\sigma$. Thus, $(X, \langle \mathcal{A} \rangle_\sigma, \mu^*)$ is a measure space and below we use the terms μ^*-null and μ^*-measurable with respect to this measure space. To be extra clear: a set $Z \subseteq X$ is μ^*-**null** if it is contained in a set $\widehat{Z} \in \langle \mathcal{A} \rangle_\sigma$ with $\mu^*(\widehat{Z}) = 0$. And a set $M \subseteq X$ is μ^*-**measurable** if $M = B \cup Z$ for some $B \in \langle \mathcal{A} \rangle_\sigma$ and μ^*-null Z.

(a) Show that for any set $Z \subseteq X$, if $\mu^*(Z) = 0$ then Z is μ^*-null. (This is not a tautology.)

(b) Similarly, prove that the σ-algebra $\mathcal{B} := \overline{\mathcal{A}}^d$ obtained in Tao's proof of Carathéodory's theorem is exactly the collection of all μ^*-measurable sets.

REMARK: One can also prove this using Borel–Cantelli.

(c) Conclude that μ^*-measurable sets are approximable by sets in $\mathcal{A}$, i.e. for each μ^*-measurable set B and $\varepsilon > 0$, there is a set $A \in \mathcal{A}$ such that $\mu^*(B \vartriangle A) < \varepsilon$.

(d) Also deduce that μ^*-measurable sets of are approximable from above by countable unions of sets in $\mathcal{A}$, i.e. for each μ^*-measurable set B and $\varepsilon > 0$, there is a countable union U of sets in $\mathcal{A}$ such that $U \supseteq B$ and $\mu^*(U) \approx_\varepsilon \mu^*(B)$.

REMARK: All these statements still hold for σ-finite premeasures, by the standard argument.

2. Prove that $\mathbb{R}^d$ does not admit a translation-invariant Borel *probability* measure. In fact, show that for any nonzero vector $\vec{t} \in \mathbb{R}^d$, there does not exist a Borel probability measure that is invariant under translation by $\vec{t}$.

HINT: Find a Borel transversal for the coset equivalence relation of the subgroup $\mathbb{Z}\vec{t} \leqslant \mathbb{R}^d$.

3. Let $\mathbb{E}_0$ be the equivalence relation on $2^{\mathbb{N}}$ of eventual equality, i.e.

$$x \mathbb{E}_0 y :\Leftrightarrow \forall^\infty n \; x(n) = y(n),$$

where $\forall^\infty n$ means for all large enough n. For each n, let $\sigma_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the n^{th} bit flip map, i.e. $\sigma_n(x)$ is the same as x except that its n^{th} coordinate is equal to $1 - x(n)$. Let Γ be the group generated by all the σ_n. (Actually, Γ is isomorphic to $\oplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$.) Then Γ naturally acts on $2^{\mathbb{N}}$.

(a) Realize that the orbit equivalence relation of this action is exactly $\mathbb{E}_0$.

(b) Note that the Bernoulli(1/2) measure is invariant under this action, i.e. for any $\mu_{1/2}$-measurable set $A \subseteq 2^{\mathbb{N}}$ and $\gamma \in \Gamma$, we have $\mu_{1/2}(\gamma A) = \mu_{1/2}(A)$.

(c) Prove that every transversal for $\mathbb{E}_0$ is not $\mu_{1/2}$-measurable.

(d) [*Optional*] Prove that for every $p \in (0,1)$, every transversal for $\mathbb{E}_0$ is not μ_p-measurable.

4. Let $(X, \mathcal{B}, \mu)$ be a measure space and recall that for a sequence (A_n) of measurable sets,

$$\limsup_n A_n := \{x \in X : \exists^\infty n\ x \in A_n\}.$$

(a) Prove the following more general (and somehow less useful) version of the Borel–Cantelli lemma: If $\mu(X) < \infty$, then $\mu(\limsup_n A_n) = \lim_N \mu(\bigcup_{n \geqslant N} A_n)$.

(b) From this version of the Borel–Cantelli lemma, deduce the versions (a) and (b) discussed in class.

5. Prove the inexistence of an uncountable almost disjoint family of non-null measurable sets for σ-finite measures.

Remark: The song that came to my mind while proving this for a finite measure was "A day in my life" by The Beatles, with the lines

And though the holes were rather small
They had to count them all

In retrospect, this summarizes most of combinatorics.

6. Fill in the gaps in the proof-sketch of Sierpinski's theorem. More precisely, let $(X, \mathcal{B}, \mu)$ be an atomless measure space and prove:

(a) Any measurable set $Y \subseteq X$ admits measurable non-null subsets of arbitrarily small measure.

Hint: Each measurable set $A \subseteq Y$ of finite measure splits into two sets of positive measure, so one of them has measure $\leqslant \frac{1}{2}\mu(A)$.

(b) [*Optional*] Let $r \in (0, \mu(X))$ and put $r' := \sup\{\mu(B) : B \in \mathcal{B}_r\}$, where $\mathcal{B}_r$ is the collection of all measurable sets of measure at most r. Prove that this supremum is achieved, i.e. there is a measurable set B with $\mu(B) = r'$. (Below the problems, I sketched two proofs.)

7. [1] We say that a real $r \in \mathbb{R}$ admits a **sequence of good rational approximations of exponent** $\alpha > 0$ if there are infinitely many pairs $(p,q) \in \mathbb{Z} \times \mathbb{N}^+$ such that

$$|r - \frac{p}{q}| < \frac{1}{q^\alpha}.$$

Dirichlet's approximation theorem (or rather its immediate consequence) states that every real admits a sequence of good rational approximations of exponent 2.

Prove however that for any $\varepsilon > 0$, almost no real admits a sequence of good rational approximations of exponent $2 + \varepsilon$, i.e., the set B of all $r \in \mathbb{R}$ that admit a sequence of good rational approximations of exponent $2 + \varepsilon$ is null (with respect to Lebesgue measure).

[1] Thanks to Robin Tucker-Drob for suggesting this question.

Hint: First, show that it is enough to prove the statement for $[0,1)$ instead of $\mathbb{R}$. Next, express B in terms of the sets

$$A_{p,q} := \left\{ r \in \mathbb{R} : |r - \frac{p}{q}| < \frac{1}{q^{2+\varepsilon}} \right\}$$

where $q \in \mathbb{N}^+$ and $p < q$ (the latter inequality can be enforced because $|r| < 1$). Finally, what is the measure of $A_{p,q}$?

8. Prove that in the 99% lemmas for Lebesgue and Bernoulli(p) measures, there are actually *arbitrarily small* boxes/cylinders whose $\geqslant$ 99% is the given measurable set. Here by *arbitrarily small*, we mean for a box that each of its dimensions are arbitrarily small, and for a cylinder that its base is arbitrarily long.

9. Follow the steps below to prove the **Steinhaus theorem**: For every Lebesgue measurable non-null set $A \subseteq \mathbb{R}^d$, the difference set $A - A := \{a_0 - a_1 : a_0, a_1 \in A\}$ contains an open neighbourhood of $\vec{0}$.

Tip: For simplicity of thought and pictures, only think of $d = 1$.

(i) Check that for all sets $U, V \subseteq \mathbb{R}^d$, we have $U \subseteq V - V$ if and only if $(V + u) \cap V \neq \emptyset$ for each $u \in U$.

(ii) Let B be a nonempty bounded open box whose at least $(1 - 1/2^{d+2})100\%$ is A. Let b_0 be the midpoint of B and put $U := B - b_0$, so U is an open box centered at $\vec{0}$. Show that for each $u \in U$, the intersection $B_u := B + u \cap B$ is a box whose each dimension is at least half of that of B, so B_u occupies at least $(1/2^d)100\%$ of B and of $B + u$.

(iii) Conclude that at least 75% of B_u is A while at least 75% of B_u is $A + u$, so $A + u \cap A \neq \emptyset$ for each $u \in U$, hence $U \subseteq A - A$.

Remark: The exact same theorem holds for the Bernoulli(1/2) measure on $2^{\mathbb{N}}$ identified with the group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. As usual, the proof is easier than for the Lebesgue measure.

10. Use the 99% lemma to prove that the equivalence relation $\mathbb{E}_0$ on $2^{\mathbb{N}}$ is $\mu_{1/2}$-ergodic.

Remark: $\mathbb{E}_0$ is actually μ_p-ergodic for all $p \in (0,1)$. Try proving it, thinking about how to overcome the fact that the action of the group $\oplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ that induces $\mathbb{E}_0$ does not preserve the Bernoulli(p) measure for $p \neq 1/2$.

Proposition (Question 6(b))**.** *Let $(X, \mathcal{B}, \mu)$ be an atomless measure space and let $r \in (0, \mu(X))$. Put $r' := \sup\{\mu(B) : B \in \mathcal{B}_r\}$, where $\mathcal{B}_r$ is the collection of all measurable sets of measure at most r. Prove that this supremum is achieved, i.e. there is a measurable set B with $\mu(B) = r'$.*

Proof 1 (transfinite measure exhaustion). Suppose towards a contradiction that there is no such set B. By Question 6(a), we can inductively build a sequence $(B_\alpha)_{\alpha < \omega_1}$ of pairwise disjoint measurable sets such that for each $\alpha < \omega_1$, the set $\bigsqcup_{\beta < \alpha} B_\beta$ has measure $< r'$. This yields a contradiction by the same pigeonhole proof of measure exhaustion as given in class, where the hypothesis of finiteness of the measure is replaced with the condition that all countable unions have measure $< r'$. $\square$

Proof 2 ($\frac{1}{2}$-trick and $\mathbb{N}$-length measure exhaustion). We inductively construct a sequence $(B_n)_{n\in\mathbb{N}}$ of pairwise disjoint measurable sets of positive measure such that for each $N \in \mathbb{N}$, the set $\bigsqcup_{n<N} B_n$ has measure $< r'$ as follows. Suppose $(B_i)_{i<n}$ is defined and let $\mathcal{B}_n$ be the collection of all measurable sets A of positive measure disjoint from $\bigsqcup_{i<n} B_i$ such that $\mu(\bigsqcup_{i<n} B_i) + \mu(A) < r'$. By Question 6(a), $\mathcal{B}_n \neq \emptyset$. Take as B_n any set in $\mathcal{B}_n$ of measure at least $\frac{1}{2}\sup\{\mu(A) : A \in \mathcal{B}_n\}$.

Then the set $B := \bigsqcup_{n\in\mathbb{N}} B_n$ has measure $\leqslant r'$ and we claim that actually $\mu(B) = r'$. Suppose otherwise, so $\mu(B) < r'$, and hence there is a measurable set A of positive measure such that $\mu(B) + \mu(A) < r'$. But then this A belongs to every $\mathcal{B}_n$, by definition, so $\mu(B_n) \geqslant \frac{1}{2}\mu(A) > 0$ for every $n \in \mathbb{N}$, contradicting $\sum_{n\in\mathbb{N}} \mu(B_n) = \mu(B) < r' < \infty$.

$\square$